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Stationary probability distribution for a particle subject to coloured noise

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Abstract. The stationary probability distribution, P_{st}, for an overdamped particle moving in a one-dimensional potential and subject to exponentially correlated noise having correlation time τ, is determined using several different methods. Firstly, the evaluation of a path-integral representation for a conditional probability distribution for small D, where D is the noise strength, is performed to show that $P_{st}(x) \sim D^{-1/2}p(x)\exp(-S(x)/D)$. The function $S(x)$ and the prefactor $p(x)$ are evaluated for various values of τ in the case of the double-well potential $V(x) = -x^2/2 + x^4/4$. Secondly, a numerical simulation of the stochastic process is carried out directly to determine the validity of the small-D approximation. Excellent agreement is found for $D \leqslant O(0.1)$, except for a small region near the top of the potential barrier at $x = 0$, when $\tau > 1$. Finally, an investigation of this region is carried out directly using a two-dimensional Fokker–Planck equation, which shows that the small-D expansion breaks down for $\tau > 1$ when $|x| \leqslant D^{1/(\tau+1)}$.

1. Introduction

The form of the stationary probability distribution, P_{st}, for a particle moving in a double-well potential subject to external (non-white) noise has still many open questions associated with it (see Doering *et al* (1989) for reviews on the topic of external noise). Even for the simplest case of an overdamped particle moving in one dimension in a potential $V(x)$ subject to exponentially correlated noise $\xi(t)$, the structure of $P_{st}(x)$ has not been completely elucidated. The purpose of this paper is to investigate this model using several different techniques (path-integral methods, numerical simulation, use of the Fokker–Planck equation near the top of the barrier) in order to understand the analytic structure of $P_{st}(x)$ more fully and to resolve various discrepancies in the results that are obtained when these different techniques are used.

The model is defined by the Langevin equation

$$\dot{x} = -V'(x) + \xi(t) \tag{1}$$

where the noise $\xi(t)$ is Gaussian with zero mean and with

$$\langle \xi(t)\xi(t')\rangle = \frac{D}{\tau}\exp(-|t-t'|/\tau). \tag{2}$$

In a recent series of papers (McKane *et al* 1990, Bray *et al* 1990, Luckock and McKane 1990) this model has been studied using a path-integral approach (an extensive list of references relating to this model is given in the first of these papers). Bray *et al* (1990) have shown that the stationary probability distribution $P_{st}(x)$ for this model has the form $P_{st}(x) \sim \exp(-S(x)/D)$ in the small-D limit. The function $S(x)$ is the action of the extremal path found by applying the method of steepest descents to the appropriate functional integral. In this paper we discuss how to extend the method of steepest descents to next order for small D. For the model described by (1) this will allow the determination of the prefactor multiplying the exponential in the expression for the stationary probability distribution.

To illustrate the method, we begin in section 2 by considering the white noise limit, $\tau = 0$, for which the stationary distribution can be obtained in closed form. In section 3 we go on to study the case of general τ. Expressions are given for the stationary joint distribution $P_{st}(\dot{x}, x)$ and the stationary marginal distribution $P_{st}(x)$ in the weak-noise limit, including the prefactors multiplying the leading-order exponential forms. In section 4 we describe the evaluation of the prefactor for the stationary marginal distribution for the special case of the quartic bistable potential, $V(x) = -x^2/2 + x^4/4$.

An important question that can reasonably be asked is, how small does D need to be for the above analysis to yield a good approximation for $P_{st}(x)$? To answer this and other points we describe the results of a numerical simulation of the Langevin equation in section 5 and compare it with the weak-noise results of section 4.

The numerical calculations of the weak-noise stationary marginal distribution show a marked change in behaviour near the top of the barrier at $\tau = 1$. In section 6 we describe how this can be studied using the Fokker–Planck equation. In particular, we determine the analytic form of $P_{st}(\dot{x}, x)$ and $P_{st}(x)$ for small x and $\dot{x}$ and discuss the anomalous behaviour of these functions for $\tau > 1$. Our conclusions are presented in section 7.

2. The white noise limit

In the limit $\tau \to 0$, the correlation function of the noise tends to a delta function,

$$\langle \xi(t)\xi(t')\rangle = 2D\delta(t - t'). \tag{3}$$

In this case a simple Fokker–Planck equation can be written down and the solution of the time independent version of this equation gives the familiar Boltzmann result $P_{st}(x) = C \exp(-V(x)/D)$, where C is a normalization constant. An analogous calculation is not possible when the noise is coloured, and it is in this situation that the path-integral approach comes into its own. In this section we will show how the Boltzmann factor can be found by evaluating a path-integral for small D when $\tau = 0$. Although this is not the most straightforward approach to determining $P_{st}(x)$ when $\tau = 0$, it is necessary to describe it in order to understand the method when $\tau \neq 0$, where it does appear to be the most efficient way of obtaining $P_{st}(\dot{x}, x)$ and hence $P_{st}(x)$.

The conditional probability distribution for the model defined by (1) and (3) can be expressed as the path-integral (Graham 1973, 1975)

$$P(x, t|x_0, t_0) = \int_{x(t_0)=x_0}^{x(t)=x} \mathcal{D}x J[x] \exp(-S[x]/D) \tag{4}$$

where

$$S[x] = \tfrac{1}{4}\int_{t_0}^{t} \mathrm{d}t(\dot{x} + V'(x))^2 \tag{5}$$

and

$$J[x] = \exp\left(\tfrac{1}{2}\int_{t_0}^{t} \mathrm{d}t\, V''(x)\right). \tag{6}$$

In the above expression for the path integral $S[x]$ is the 'action' and $J[x]$ is the Jacobian associated with the functional change of variables from ξ to x in (1). Evaluating the path-integral (4) for small D by the method of steepest descents gives

$$P(x, t|x_0, t_0) \simeq NJ[x_c](\mathrm{Det}\, M)^{-1/2} \exp(-S[x_c]/D) \tag{7}$$

where x_c is the extremal path which minimizes the action $S[x]$, M is the operator associated with Gaussian fluctuations around the extremal path, and N is a normalization constant. We shall take x_0 to be the position of a local minimum of the potential since this is relevant to the calculation of the stationary probability distribution defined by

$$P_{\mathrm{st}}(x) \equiv \lim_{t_0 \to -\infty} P(x, t|x_0, t_0). \tag{8}$$

The extremal condition $\delta S[x]/\delta x = 0$ leads to the equations $\dot{x}_c = \pm V'(x_c)$. For the uphill path beginning at a local minimum, the positive sign is appropriate so that

$$\dot{x}_c = V'(x_c). \tag{9}$$

The operator M is proportional to the second functional derivative of the action which is obtained from (5) as

$$M = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V''(x_c)^2 + V'(x_c)\, V'''(x_c). \tag{10}$$

We shall assume to start with that $V(x)$ is a single-well potential with its minimum at x_0 since this is the simplest case. However, we shall go on to show how the analysis may be extended to the case where $V(x)$ has more than one minimum.

The stationary probability distribution is obtained by taking the limit $t_0 \to -\infty$ in (4). However, because some quantities in which we will be interested, diverge in this limit, to begin with we will assume that t_0 is large and negative and only in the final stage of our calculation will we let $t_0 \to -\infty$.

We can easily find expressions for the action and the Jacobian using equation (9). For the action we have

$$S[x_c] = \tfrac{1}{4}\int_{t_0}^{t} \mathrm{d}t(\dot{x}_c + V'(x_c))^2 = V(x) - V(x_0) \tag{11}$$

while for the Jacobian

$$J[x_c] = \exp\left(\tfrac{1}{2}\int_{t_0}^{t} \mathrm{d}t\, V''(x_c)\right) = \exp\left(\tfrac{1}{2}\int_{t_0}^{t} \mathrm{d}t \frac{1}{\dot{x}_c}\frac{\mathrm{d}}{\mathrm{d}t}\dot{x}_c\right) = \frac{\dot{x}_c(t)^{1/2}}{\dot{x}_c(t_0)^{1/2}}. \tag{12}$$

We will now turn our attention to the calculation of the determinant. Since M given by (10) is a Hermitian second-order differential operator it can be shown that (Coleman 1979)

$$\mathrm{Det}\, M \propto \phi(t) \tag{13}$$

where $\phi(t)$ is the most general solution to the linear differential equation

$$M\phi(t) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V''^2(x_c) + V'(x_c)\, V'''(x_c)\right)\phi(t) = 0 \tag{14}$$

subject to the boundary conditions

$$\phi(t_0)=0 \qquad \dot{\phi}(t_0)=1. \tag{15}$$

One solution to (14), by virtue of the Euler–Lagrange equation (9), is $\phi(t)=\dot{x}_c(t)$. Making the substitution $\phi=\dot{x}_c h$ into (14), where h is some function of t, gives

$$\phi(t)=A\dot{x}_c(t)\dot{x}_c^2(t_0)\int_{t_0}^{t}\mathrm{d}t\left(\frac{1}{\dot{x}_c^2(t)}\right)+B\dot{x}_c(t). \tag{16}$$

Here A and B are constants which are determined by the boundary conditions (15) to be

$$A=1/\dot{x}_c(t_0) \qquad B=0. \tag{17}$$

Therefore one obtains the following expression for the determinant:

$$\operatorname{Det} M\propto\dot{x}_c(t)\dot{x}_c(t_0)\int_{t_0}^{t}\mathrm{d}t\left(\frac{1}{\dot{x}_c^2(t)}\right). \tag{18}$$

At large negative times, x_c is very close to the position of the minimum of the potential and, since $\dot{x}_c=V'(x_c)$, $\dot{x}_c$ is very close to zero. Therefore we expect the integral in (18) to be dominated by the integrand for values of t close to t_0, leading to

$$\int_{t_0}^{t}\mathrm{d}t\left(\frac{1}{\dot{x}_c^2(t)}\right)\simeq\frac{k}{\dot{x}_c^2(t_0)} \tag{19}$$

where k is some positive constant. Therefore the final result for the determinant is

$$\operatorname{Det} M\propto\frac{\dot{x}_c(t)}{\dot{x}_c(t_0)}. \tag{20}$$

Now the various terms contributing to (7) have been dealt with. Inserting the results (11), (12) and (20) into (7) we see that the prefactor, $NJ[x_c](\operatorname{Det} M)^{-1/2}$, is constant in the limit $t_0\to-\infty$ and so the first-order small D approximation to the stationary probability distribution is just the Boltzmann factor,

$$P_{st}(x)=C\exp(-V(x)/D) \tag{21}$$

where the constant C is determined by normalizing the probability distribution. Thus we have demonstrated, in the case of white noise, how the exact result for the stationary probability distribution found by solving the time-independent Fokker–Planck equation may also be found by using a path-integral approach. We have only considered the effects of fluctuations to first order; the mechanism that causes all higher order corrections to vanish is not clear from the analysis presented here.

To extend the above method to a potential with more than one minimum is straightforward. As an example, consider the double-well potential represented in figure 1 with two minima at a and c and a single maximum at $x=b$. The point $x=b$ is a singular point of (9) in the following sense: near the top of the barrier, for values of x close to b it is possible to expand $V(x)$ in (9) as a Taylor series about $x=b$ to obtain the linearized equation,

$$\dot{x}_c\simeq-\alpha(x_c-b) \tag{22}$$

where $\alpha\equiv-V''(b)$. For paths in the right-hand well, i.e. for $x_c>b$, the solution of (22) is

$$x_c=b+A\exp(-\alpha t) \tag{23}$$

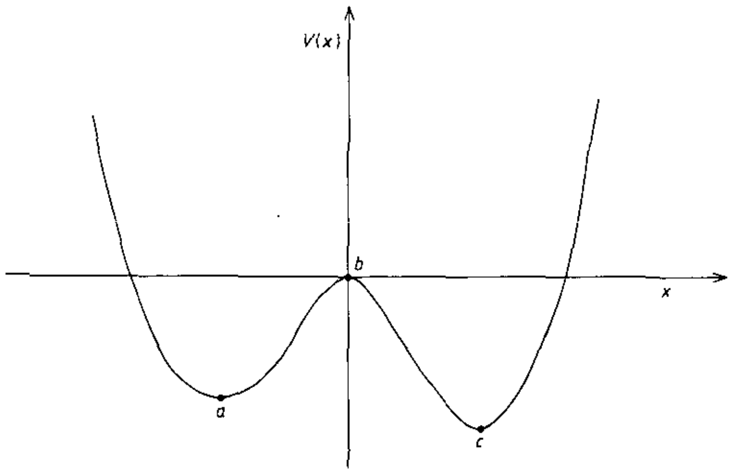

Figure 1. A typical double-well potential.

where A is some positive constant. For finite t, (23) can be satisfied for x_c arbitrarily close to b, but for $x_c = b$, (23) can be satisfied only in the limit $t \to \infty$. Thus there are no extremal paths defined on the semi-infinite time interval $(-\infty, t)$ which begin at the minimum of a potential and end exactly at the position of a neighbouring maximum. A corollary to this statement is that there are no paths on $(-\infty, t)$ linking two separate wells. Therefore to obtain the stationary probability distribution, within the small D approximation, one can treat each well independently.

Applying the result (21) to each well of the double-well potential separately, one obtains

$$P_{st}(x) = \begin{cases} M_1 \exp(-V(x)/D) & \text{if } -\infty < x < b \\ M_2 \exp(-V(x)/D) & \text{if } b < x < \infty. \end{cases} \tag{24}$$

For $P_{st}(x)$ to be continuous at b we must have that

$$\lim_{x \to b-} P_{st}(x) = P_{st}(b) = \lim_{x \to b+} P_{st}(x) \tag{25}$$

and hence that $M_1 = M_2$. Thus

$$P_{st}(x) = M_1 \exp(-V(x)/D) \tag{26}$$

for all values of x. Of course, (26) may again be obtained much more simply as the solution of the time-independent Fokker–Planck equation. We stress again that we have rederived the result using the method of steepest descents in preparation for the next section where we consider the case of non-zero τ for which the Fokker–Planck equation cannot be solved in closed form.

The mean time of escape from a domain of attraction for this model has been calculated to leading order using the method of steepest descents by Bray *et al* (1990). However, obtaining the prefactor multiplying the exponential part of the first passage time is in general more difficult than obtaining the prefactor for the stationary probability distribution. To see this, we again consider the white noise model with the double-well potential shown in figure 1. To leading order, the escape time from the left-hand well to the right-hand well is just given by the Arrhenius result,

$$\bar{T} \sim \exp\left(\frac{V(b)-V(a)}{D}\right). \tag{27}$$

The uphill extremal path in this case is again given by (9) but this time the boundary conditions are $x_c(-\infty)=a$ and $x_c(\infty)=b$. If we want to go beyond leading order, we have to evaluate both $J[x_c]$ and Det M in the expression for the conditional probability distribution (7). The evaluation of the Jacobian factor proceeds as before, but the calculation of the determinant presents a problem. According to (14), $M\dot{x}_c=0$, and because of the boundary conditions of this problem, $\dot{x}_c$ vanishes at times $t=\pm\infty$. Hence $\dot{x}_c$ is an eigenfunction of M with eigenvalue zero. Since the determinant is proportional to the product of the eigenvalues this means that it will vanish and hence that the conditional probability given by (7) will diverge in the limit $t_0\to-\infty$, $t\to\infty$. Fortunately, this divergence can be taken care of by using the method of collective coordinates. However, this does introduce additional complexity into the calculation, and the prefactor for the mean first passage time has so far only been obtained in a small τ-expansion (Klosek-Dygas *et al* 1989, Luckock and McKane 1990).

3. Coloured noise

We now consider the case $\tau\neq 0$, where the Fokker–Planck equation which is equivalent to (1) and (2) cannot be solved in closed form for the stationary probability distribution. The conditional probability distribution can, however, be expressed as the path integral (McKane *et al* 1990)

$$P(\dot{x}, x, t\,|\,\dot{x}_0, x_0, t_0)=\int_{x(t_0)=x_0,\dot{x}(t_0)=\dot{x}_0}^{x(t)=x,\dot{x}(t)=\dot{x}} \mathscr{D}x J[x]\exp(-S[x]/D) \tag{28}$$

with

$$J[x]=\exp\left(\tfrac{1}{2}\int_{t_0}^{t}\mathrm{d}t(\tau^{-1}+V''(x))\right) \tag{29}$$

and

$$S[x]=\tfrac{1}{4}\int_{t_0}^{t}\mathrm{d}t[(\dot{x}+V')+\tau(\ddot{x}+\dot{x}V'')]^2. \tag{30}$$

Evaluating (28) by the method of steepest descents gives

$$P(\dot{x}, x, t\,|\,\dot{x}_0, x_0, t_0)\simeq NJ[x_c](\mathrm{Det}\, M)^{-1/2}\exp(-S(\dot{x},x)/D) \tag{31}$$

where $S(\dot{x},x)$ is the action of the extremal path, x_c, which satisfies the Euler–Lagrange equation (Bray *et al* 1990)

$$-\ddot{x}+V'V''+\tau^2[\ddddot{x}+3\ddot{x}\dot{x}V'''+\dot{x}^3V''''-\dot{x}^2V''V'''-\ddot{x}V''^2]=0. \tag{32}$$

Multiplying the above equation by $\dot{x}$ and integrating with respect to t gives

$$\dot{x}^2-V'^2=\tau^2[2\dddot{x}\dot{x}-\ddot{x}^2+2\dot{x}^3V'''-\dot{x}^2V''^2] \tag{33}$$

where the integration constant vanishes if the initial conditions are set in the infinitely distant past. The second functional derivative of the action, M, is obtained from (30) as (Luckock and McKane 1990)

$$M=-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\gamma(t)+\tau^2\left[\frac{\mathrm{d}^4}{\mathrm{d}t^4}+\alpha(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\dot{\alpha}(t)\frac{\mathrm{d}}{\mathrm{d}t}+\beta(t)\right] \tag{34}$$

where

$$\begin{aligned}
\alpha(t)&=3\dot{x}_c V_c'''-(V_c'')^2\\
\beta(t)&=3\dot{x}_c\ddot{x}_c V_c''''+(\dot{x}_c)^3V_c'''''-2\ddot{x}_c V_c'' V_c'''-(\dot{x}_c)^2(V_c''')^2-(\dot{x}_c)^2 V_c'' V_c''''\\
\gamma(t)&=V_c' V_c'''+(V_c'')^2.
\end{aligned} \tag{35}$$

Here we have introduced the notation $V_c'\equiv V'(x_c)$, $V_c''\equiv V''(x_c)$, etc.

The stationary probability distribution is obtained in the usual way by setting the initial conditions in the infinitely distant past. For simplicity, to start with we will assume that $V(x)$ has a single minimum at $x=x_0$. For this case $S(\dot{x},x)$ has a global minimum at $(\dot{x}=0, x=x_0)$.

To determine the normalization of the stationary probability distribution in the small-D limit we need only consider the region of x and $\dot{x}$ about the global minimum $(\dot{x}=0, x=x_0)$ of the action, since for small D the stationary joint probability distribution will be sharply peaked about this point. For values of x near x_0, the potential will be approximately quadratic with

$$V(x)\simeq\frac{V_0''}{2}(x-x_0)^2 \tag{36}$$

where $V_0''\equiv V''(x_0)$. The stationary probability distribution $P_0(\dot{x},x)$ for a quadratic potential can obtained exactly by solving the time-independent Fokker–Planck equation which, using the form (36) for the potential, is

$$0=-\frac{\partial}{\partial x}(\dot{x}P_0)+\frac{\partial}{\partial\dot{x}}\left([V_0''+\tau^{-1}][\dot{x}P_0]+V_0''\tau^{-1}(x-x_0)P_0+\frac{D}{\tau^2}\frac{\partial P_0}{\partial\dot{x}}\right). \tag{37}$$

The solution of which is

$$P_0(\dot{x},x)=\frac{(V_0''\tau)^{1/2}(1+V_0''\tau)}{2\pi D}\exp\left(-\frac{1+V_0''\tau}{2D}(V_0''(x-x_0)^2+\tau\dot{x}^2)\right). \tag{38}$$

Since the potential is approximately quadratic for x near x_0, the stationary probability distribution in the small-D limit will be given by (38) in this region, and in particular we will have that

$$P_{st}(0,x_0)=P_0(0,x_0)=\frac{(V_0''\tau)^{1/2}(1+V_0''\tau)}{2\pi D}. \tag{39}$$

On the other hand, (31) gives the stationary joint probability distribution at $\dot{x}=0$, $x=x_0$ as

$$P_{st}(0,x_0)\equiv P(0,x_0,t\,|\,0,x_0,-\infty)=\lim_{t_0\to-\infty}NJ_0(\mathrm{Det}\,M_0)^{-1/2}. \tag{40}$$

Here, J_0 and M_0 are the Jacobian and operator associated with fluctuations about the minimum of the potential well, where the potential is quadratic with $V(x)$ given by (36). Using (36) in (29) and (34), one finds

$$J_0 = \exp\left(\tfrac{1}{2}\int_{t_0}^{t} \mathrm{d}t(\tau^{-1} + V_0'')\right) = \exp\left(\frac{T}{2}(\tau^{-1} + V_0'')\right) \tag{41}$$

where $T \equiv t - t_0$, and

$$M_0 = \tau^2 \frac{\mathrm{d}^4}{\mathrm{d}t^4} - (1 + (\tau V_0'')^2)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (V_0'')^2. \tag{42}$$

Comparing (39) and (40) gives the value of the normalization constant in (31) as

$$N = \frac{(V_0''\tau)^{1/2}(1 + V_0''\tau)}{2\pi D} \lim_{t_0 \to -\infty} [J_0^{-1}(\mathrm{Det}\, M_0)^{1/2}] \tag{43}$$

and therefore

$$P_{\mathrm{st}}(\dot{x}, x) \equiv P(\dot{x}, x, t\,|\,0, x_0, -\infty) \tag{44}$$

$$= \left[\frac{J[x_c](\mathrm{Det}\, M)^{-1/2}}{J_0(\mathrm{Det}\, M_0)^{-1/2}}\right] \frac{(V_0''\tau)^{1/2}(1 + V_0''\tau)}{2\pi D} \exp\left(-\frac{S(\dot{x}, x)}{D}\right) \tag{45}$$

where the term in square brackets is to be evaluated in the limit $t_0 \to -\infty$.

For general τ the Euler–Lagrange equation (33) has to be solved numerically for a given potential $V(x)$ in order to find $x_c(t)$. It is then a straightforward task to obtain $S(\dot{x}, x)$ and $J[x_c]$ in (45) by substituting $x_c(t)$ into the expressions (30) and (29) respectively.

The ratio Det M/Det M_0 in (45) has also to be found numerically. To deal with the determinants we will use a result by Dreyfus and Dym (1978). These authors consider the case of an nth order linear differential operator $M(n)$ where n is a positive integer. When n is even they show that one can reduce the problem of obtaining the determinant of $M(n)$ to that of finding $n/2$ linearly independent solutions to the initial value problem $M(n)\phi = 0$. For the case $n = 2$, their result coincides with the result (13). For $n = 4$, their theorem can be stated in the following way.

Let M_1 and M_2 be two fourth-order operators acting on functions defined in the interval (t_0, t_1) and normalized so that the coefficient of $\mathrm{d}^4/\mathrm{d}t^4$ is the same in both operators. If the eigenfunctions of M_1 and M_2 and the first derivatives of the eigenfunctions are set to be zero at $t = t_0$ and $t = t_1$ then

$$\frac{\mathrm{Det}\, M_1}{\mathrm{Det}\, M_2} = \frac{\mathrm{Det}\, L_1}{\mathrm{Det}\, L_2} \tag{46}$$

where

$$L_i \equiv \begin{pmatrix} u_1^i(t_1) & u_2^i(t_1) \\ \dot{u}_1^i(t_1) & \dot{u}_2^i(t_1) \end{pmatrix} \qquad i = 1, 2 \tag{47}$$

and the $\{u_j^i\}$ satisfy

$$M_i u_j^i(t) = 0 \qquad i, j = 1, 2 \tag{48}$$

with the initial conditions

$$u_1^i(t_0) = 0 \qquad \dot{u}_1^i(t_0) = 0 \qquad \ddot{u}_1^i(t_0) = 1 \qquad \dddot{u}_1^i(t_0) = 0 \qquad i = 1, 2 \tag{49}$$

and

$$u_2^i(t_0)=0 \qquad \dot{u}_2^i(t_0)=0 \qquad \ddot{u}_2^i(t_0)=0 \qquad \dddot{u}_2^i(t_0)=1 \qquad i=1,2. \tag{50}$$

We will use the result (46) to calculate the ratio

$$\frac{\text{Det } M}{\text{Det } M_0} \tag{51}$$

in (45) on the time interval (t_0, t_1). We will deal first with the operator M_0 given by (42). We are to find two solutions to the homogeneous equation

$$M_0 u(t) \equiv \left[\tau^2 \frac{\mathrm{d}^4}{\mathrm{d}t^4} - (1+(\tau V_0'')^2)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (V_0'')^2\right] u(t) = 0 \tag{52}$$

subject to the two sets of boundary conditions (49) and (50). In this case the homogeneous equation can be solved analytically yielding the solutions

$$u_1(t) = \frac{\tau^2}{1-(V_0'')^2\tau^2}[\cosh((t-t_0)/\tau) - \cosh(V_0''(t-t_0))] \tag{53}$$

and

$$u_2(t) = \frac{\tau^2}{1-(V_0'')^2\tau^2}\left[\tau \sinh((t-t_0)/\tau) - \frac{1}{V_0''}\sinh(V_0''(t-t_0))\right]. \tag{54}$$

Now we construct the matrix

$$L_0 \equiv \begin{pmatrix} u_1(t_1) & u_2(t_1) \\ \dot{u}_1(t_1) & \dot{u}_2(t_1) \end{pmatrix} \tag{55}$$

and calculate its determinant. Using (53) and (54) one finds

$$\text{Det } L_0 = \frac{\tau^3 \Delta(t_1 - t_0)}{V_0''(1-(V_0'')^2\tau^2)^2} \tag{56}$$

where

$$\Delta(T) = (1+(V_0'')^2\tau^2)\sinh(V_0''T)\sinh(T/\tau) + 2V_0''\tau(1-\cosh(V_0''T)\cosh(T/\tau)). \tag{57}$$

Repeating the above procedure for the operator M given by (34), we obtain the determinant Det L where

$$L \equiv \begin{pmatrix} v_1(t_1) & v_2(t_1) \\ \dot{v}_1(t_1) & \dot{v}_2(t_1) \end{pmatrix} \tag{58}$$

and v_1 and v_2 are both solutions of

$$Mv(t) = 0. \tag{59}$$

To obtain v_1 and v_2, (59) has to be solved numerically, subject to the same boundary conditions as u_1 and u_2 in (49) and (50).

So, in summary, from (46) we have that

$$\frac{\text{Det } M}{\text{Det } M_0} = \frac{\text{Det } L}{\text{Det } L_0} \tag{60}$$

where Det L_0 is given by (56), and L is the two by two matrix (58) whose entries are determined by solving (59) numerically.

Using (60), the term enclosed by square brackets in (45) can be written as

$$\frac{J[x_c](\mathrm{Det}\ M)^{-1/2}}{J_0(\mathrm{Det}\ M_0)^{-1/2}}=\frac{J[x_c](\mathrm{Det}\ L)^{-1/2}}{J_0(\mathrm{Det}\ L_0)^{-1/2}}. \tag{61}$$

In the limit $t_0\to-\infty$ the denominator on the right-hand side of (61) is found, using (41) and (56), to be

$$J_0(\mathrm{Det}\ L_0)^{-1/2}=\frac{2}{\tau}\left(\frac{V_0''}{\tau}\right)^{1/2}(1+V_0''\tau). \tag{62}$$

Substituting (61) and (62) into (45) we at last obtain the normalized stationary joint probability distribution as

$$P_{\mathrm{st}}(\dot{x},x)=\frac{\tau^2}{4\pi D}J[x_c](\mathrm{Det}\ L)^{-1/2}\exp(-S(\dot{x},x)/D). \tag{63}$$

We now consider the marginal stationary probability distribution $P_{\mathrm{st}}(x)$, which is obtained by integrating out the velocity $\dot{x}$ from the above expression. One can show that for fixed x the action $S(\dot{x},x)$ has a single minimum with respect to $\dot{x}$ at $\dot{x}=0$. Expanding the action as a Taylor series about $\dot{x}=0$ we obtain,

$$S(\dot{x},x)=S(0,x)+\frac{\dot{x}^2}{2}\left.\frac{\partial^2S(\dot{x},x)}{\partial\dot{x}^2}\right|_{\dot{x}=0}+\ldots \tag{64}$$

where the term linear in $\dot{x}$ is zero because the action is stationary. Using (64) in (63), the integral over $\dot{x}$ can be done by steepest descents to obtain the marginal stationary probability distribution,

$$P_{\mathrm{st}}(x)=\tau^2(8\pi D)^{-1/2}\left(\left.\frac{\partial^2S(\dot{x},x)}{\partial\dot{x}^2}\right|_{\dot{x}=0}\right)^{-1/2}J[x_c](\mathrm{Det}\ L)^{-1/2}\exp(-S(0,x)/D). \tag{65}$$

4. Numerical calculation in the small-D limit

In this section we describe the computation of the marginal stationary probability distribution $P_{\mathrm{st}}(x)$ for the model described by (1) and (2) in the small-D limit. We will present numerical results for the quartic double-well potential,

$$V(x)=-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4. \tag{66}$$

In view of the discussion at the end of section 2, it is clear that we are able to treat each well separately. Moreover, we need only consider the right-hand potential well, $0<x<\infty$, containing the minimum $x=1$ sunce $V(x)$, and therefore $P_{\mathrm{st}}(x)$, is symmetric about $x=0$. Note that in this case, the potential has two wells which contribute equal amounts to the area under the probability distribution curve. Therefore, the normalization of $P_{\mathrm{st}}(x)$ given in (65) is halved when we consider only one well, i.e.

$$P_{\mathrm{st}}(x)=\frac{\tau^2}{2}(8\pi D)^{-1/2}\left(\left.\frac{\partial^2S(\dot{x},x)}{\partial\dot{x}^2}\right|_{\dot{x}=0}\right)^{-1/2}J[x_c](\mathrm{Det}\ L)^{-1/2}\exp(-S(0,x)/D). \tag{67}$$

We outline below, for the potential given by (66), the procedure to be followed in evaluating numerically the above expression for the marginal stationary probability distribution.

1. First one must solve the Euler–Lagrange equation (33) to obtain the extremal path $x_c(t)$. If the uphill path starting at the minimum $x_c=1$ and ending at the point $x_c=x$, where $0<x<\infty$, is defined on the semi-infinite time interval $(-\infty, 0)$, then the appropriate boundary conditions are $x_c(-\infty)=1$, $x_c(0)=x$ and $\dot{x}_c(0)=0$. However, in practice it is only possible to obtain the uphill solution on the *finite* time interval $(-T, 0)$. The boundary condition at time $-T$ where T is large can be obtained by linearizing (32) for values of x close to $x=1$, where V is approximately quadratic. Doing this, one finds that the solution of (32) for large negative times obeys

$$\tau\ddot{x}_c-(1+V_0''\tau)\dot{x}_c+V_0''(x_c-x_0)=0. \tag{68}$$

For a given value of x and a large but finite value of T, we solved (33) numerically on $(-T, 0)$ subject to the condition (68) at $t=-T$ and the conditions $x_c=x$, $\dot{x}_c=0$ at $t=0$. The software package used to solve (33) was COLSYS (Ascher *et al* 1981).

2. Having obtained the extremal path $x_c(t)$, the action $S(0, x)$ and the Jacobian $J[x_c]$ are easily found by evaluating numerically the integrals in (30) and (29).

3. The second partial derivative in (67) can be approximated thus:

$$\left.\frac{\partial^2 S(\dot{x}, x)}{\partial \dot{x}^2}\right|_{\dot{x}=0} \simeq \frac{S(\delta, x)+S(-\delta, x)-2S(0, x)}{\delta^2} \tag{69}$$

for small δ. $S(y, x)$ can be obtained for any y by solving (33) on $(-T, 0)$, subject to the condition (68) at $t=-T$ and $x_c=x$, $\dot{x}_c=y$ at $t=0$, and then substituting the solution into (30). Hence the right-hand side of (69) can be evaluated numerically for any value of δ. In our calculation of the second derivative we chose δ to be typically of the order 10^{-4}.

4. We saw in the last section how to calculate Det L. We recall that one must first find two linear solutions $v_1(t)$ and $v_2(t)$, defined on $(-T, 0)$, to the homogeneous equation $Mv=0$ subject to the initial conditions

$$v_1(-T)=0 \qquad \dot{v}_1(-T)=0 \qquad \ddot{v}_1(-T)=1 \qquad \dddot{v}_1(-T)=0 \tag{70}$$

and

$$v_2(-T)=0 \qquad \dot{v}_2(-T)=0 \qquad \ddot{v}_2(-T)=0 \qquad \dddot{v}_2(-T)=1. \tag{71}$$

The operator M is given by (34). The differential equation $Mv=0$ has a fairly complicated form given that its coefficients depend on the numerical solution, $x_c(t)$, to another differential equation (33). However, because M is linear, the solutions v_1 and v_2 are unique and can be obtained numerically quite easily. Having found $v_1(t)$ and $v_2(t)$, we then form the matrix

$$L\equiv\begin{pmatrix} v_1(0) & v_2(0) \\ \dot{v}_1(0) & \dot{v}_2(0) \end{pmatrix} \tag{72}$$

and calculate its determinant.

5. The marginal stationary probability distribution can now be obtained from (67), since all the various terms in (67) have been dealt with above.

The numerical results for various values of τ are displayed in figures 2–5. Figure 2 shows the action $S(0, x)$, which had been previously obtained by Bray *et al* (1990). Figure 3 shows the second derivative $\partial^2 S(\dot{x}, x)/\partial \dot{x}^2|_{\dot{x}=0}$, found using step 2 of the above procedure, and in figure 4 the product $J[x_c](\mathrm{Det}\, L)^{-1/2}$ is plotted using steps 2 and 4. Figure 5 is a graph of the prefactor multiplying the exponential in (67), which is obtained by combining the results displayed in figures 3 and 4.

The numerical results indicate that for $\tau > 1$ the prefactor approaches zero as x approaches zero, while for $\tau \leqslant 1$ the prefactor is non-zero for all x. In section 6 we present analytical results which explain this observation. First, however, we shall compare the results of this section with direct numerical simulations of the Langevin equation.

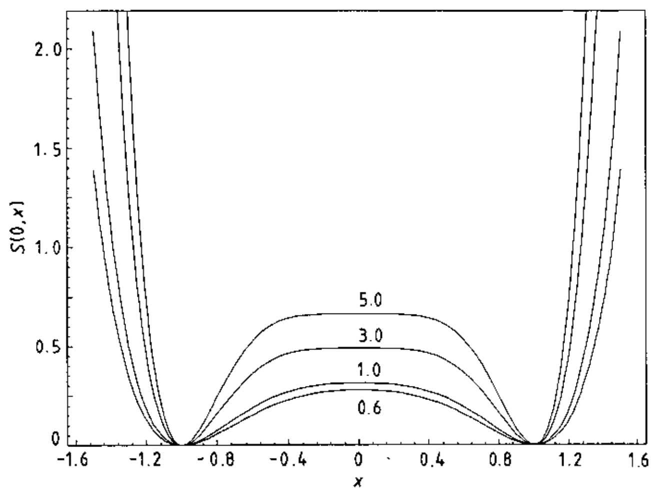

Figure 2. The minimal action $S(0, x)$ plotted for the values of $\tau = 0.6$, 1.0, 3.0 and 5.0.

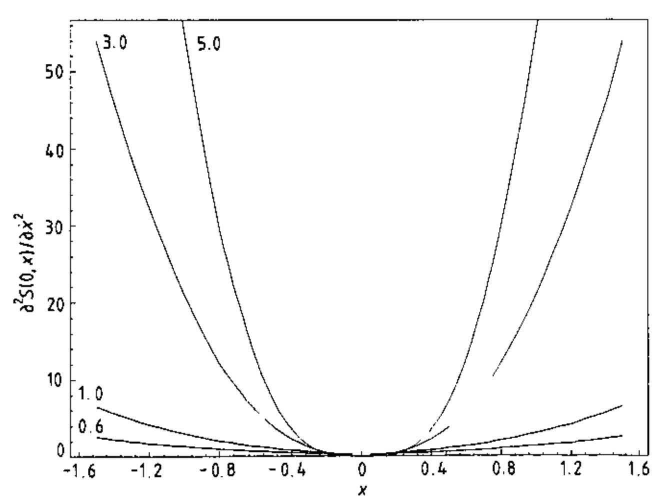

Figure 3. The second derivative $\partial^2 S(0, x)/\partial \dot{x}^2$ for $\tau = 0.6$, 1.0, 3.0 and 5.0.

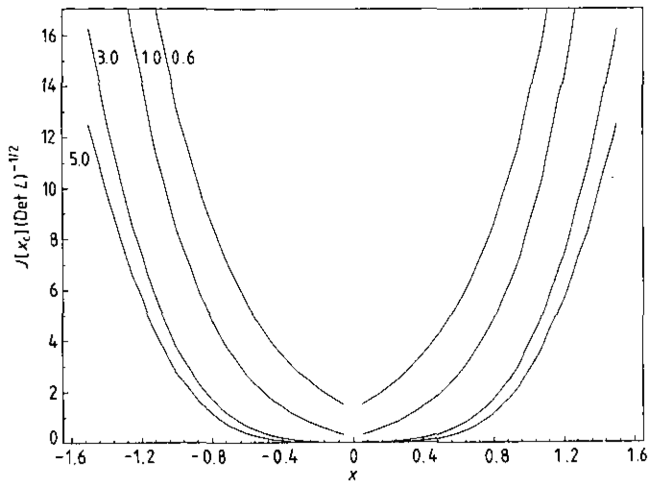

Figure 4. The quantity $J[x_c](\mathrm{Det}\ L)^{-1/2}$, which occurs in (67), for $\tau = 0.6$, 1.0, 3.0 and 5.0.

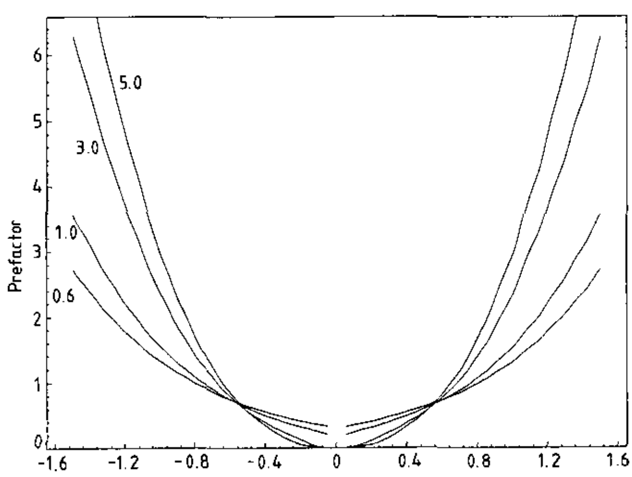

Figure 5. The prefactor multiplying the exponential part of $P_{st}(x)$, for $\tau = 0.6$, 1.0, 3.0 and 5.0.

5. Numerical simulations

We carried out numerical simulations on the system of equations

$$\dot{x} = -V'(x) + \xi \qquad \dot{\xi} = -\frac{1}{\tau}\xi + \frac{1}{\tau}\eta(t) \tag{73}$$

where $\eta(t)$ is Gaussian, delta-correlated noise with diffusion constant D and $V(x)$ is the quartic bistable potential (66). This system of equations is equivalent to the coloured noise Langevin equation (1) if the initial condition on $\xi(t)$ is set in the infinitely distant past.

We used a second order Runge-Kutta method to evolve the equations and the Box-Muller algorithm (Knuth 1969) to generate the Gaussian stochastic variables, η, from uniform stochastic variables. After evolving the equations for a sufficiently long time, dependence on the initial conditions is lost. We evolved the equations for 100 000 different realizations of the noise, $\eta(t)$, after which the stationary distribution $P_{st}(x)$ was extracted by dividing up the x-axis into sections and counting the number of simulations which resulted in the final value of the stochastic variable $x(t)$ ending up in a particular section.

To compare the results of the simulation with that of the small-D analysis of the previous section we extract from the stationary probability distribution the quantity

$$G(x) = 2(8\pi D)^{1/2}\tau^{-2}\exp(S(0,x)/D)P_{st}(x) \tag{74}$$

where $S(0,x)$ is the action which was calculated in the previous section. Comparing (74) and (67), we see that the function $G(x)$ in the steepest descent approximation is just the D-independent part of the prefactor which we shall denote by $G_{D=0}(x)$:

$$G_{D=0}(x) = \left(\frac{\partial^2 S(\dot{x},x)}{\partial\dot{x}^2}\bigg|_{\dot{x}=0}\right)^{-1/2} J[x_c](\mathrm{Det}\, L)^{-1/2}. \tag{75}$$

We shall compare this quantity with $G_{\mathrm{Sim}}(x)$ which is calculated from (74) using the stationary probability distribution $P_{st}(x)$ determined directly from the numerical simulations. The results for various values of D are shown in figures 6–8 for $\tau=3$. From these graphs we see that the small-D analysis gives results which are in excellent agreement with the numerical simulations for $D \leqslant O(0.1)$.

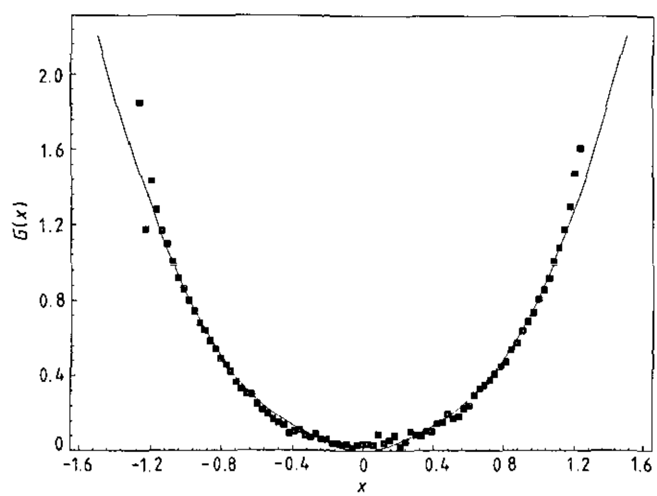

Figure 6. The prefactor $G(x)$ plotted for $D=0.1$ and $\tau=3$. The points are from the numerical simulations and the line is from the path-integral analysis.

In order to look more closely at the region near the top of the potential barrier, where the probability distribution is at a local minimum, we have plotted $\ln P_{st}(x)$ against x in figure 9 for $D=0.1$. The graph shows that the predictions of the theory hold well for $\tau=0.6$ at all values of x but that for $\tau=3$ the theory breaks down when x is very close to zero. In fact, as we shall show in the following section, there is a threshold value of τ, $\tau=1$, above which the small-D analysis predicts a vanishing stationary probability distribution at $x=0$.

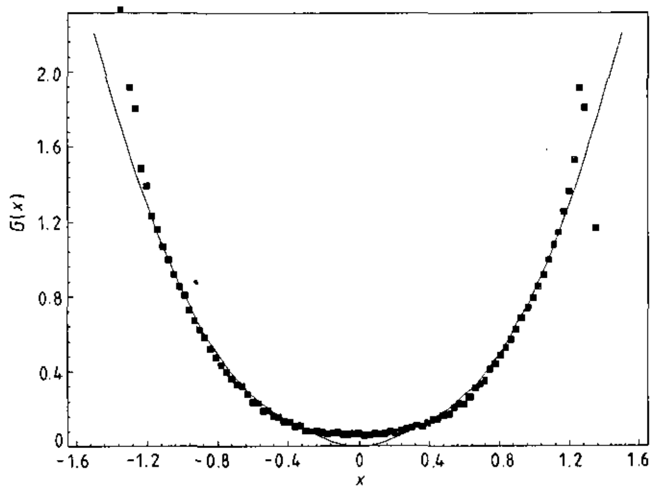

Figure 7. The prefactor $G(x)$ plotted for $D=0.2$ and $\tau=3$. The points are from the numerical simulations and the line is from the path-integral analysis.

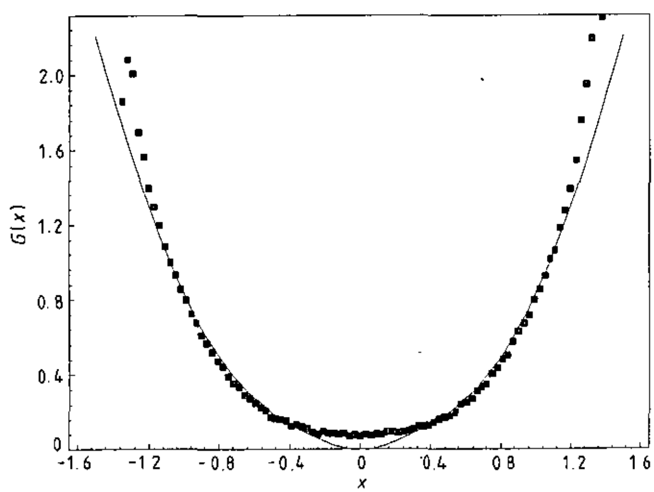

Figure 8. The prefactor $G(x)$ plotted for $D=0.3$ and $\tau=3$. The points are from the numerical simulations and the line is from the path-integral analysis.

We note that the results of the simulations presented here are in agreement with the numerical solution of the Fokker–Planck equation obtained by Jung *et al* (1989) using the matrix continued-fraction method.

6. Small-D analysis near the top of the potential barrier

In this section, we shall discuss how, for small D near the top of the barrier at $x=0$, analytic results for $P_{st}(x)$ may be obtained directly from the two-dimensional Fokker–Planck equation. This will enable us to investigate the disagreement between the predictions made on the basis of steepest descent calculations and the numerical simulations that were described at the end of the last section.

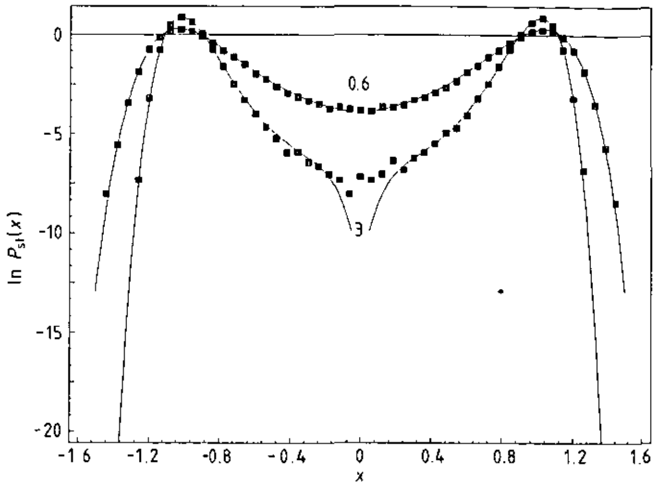

Figure 9. The value of ln $P_{st}(x)$ plotted for both $\tau=0.6$ and $\tau=3$ at $D=0.1$. The points are from the numerical simulations and the line is from the path-integral analysis.

The time-independent Fokker–Planck equation equivalent to (1) and (2) is (McKane *et al* 1990)

$$0=-\dot{x}\frac{\partial P}{\partial x}+\frac{1}{\tau}(1+\tau V''(x))\frac{\partial}{\partial \dot{x}}(\dot{x}P)+\frac{V'(x)}{\tau}\frac{\partial P}{\partial \dot{x}}+\frac{D}{\tau^2}\frac{\partial^2 P}{\partial \dot{x}^2}. \tag{76}$$

For small D we seek a solution of the form,

$$P_{st}(\dot{x},x)=N\exp(-f(\dot{x},x)/D) \tag{77}$$

where $f(\dot{x},x)$ can be expanded as a power series in D:

$$f(\dot{x},x)=f_0(\dot{x},x)+Df_1(\dot{x},x)+D^2f_2(\dot{x},x)+\dots. \tag{78}$$

Keeping only the first two terms of this expansion gives us the form

$$P_{st}(\dot{x},x)=[N\exp(-f_1(\dot{x},x))]\exp(-f_0(\dot{x},x)/D) \tag{79}$$

Comparing this with (63), we see that the function $f_0(\dot{x},x)$ can be identified with the action $S(\dot{x},x)$, and the term in square brackets with the prefactor. Substituting the above form into (76), and equating powers of D, gives the following equations in f_0 and f_1:

$$0=\dot{x}\frac{\partial f_0}{\partial x}-\frac{1}{\tau}(1+\tau V'')\dot{x}\frac{\partial f_0}{\partial \dot{x}}-\frac{V'}{\tau}\frac{\partial f_0}{\partial \dot{x}}+\frac{1}{\tau^2}\left(\frac{\partial f_0}{\partial \dot{x}}\right)^2 \tag{80}$$

and

$$0=\dot{x}\frac{\partial f_1}{\partial x}+\frac{1}{\tau}(1+\tau V'')\left(1-\dot{x}\frac{\partial f_1}{\partial \dot{x}}\right)-\frac{V'}{\tau}\frac{\partial f_1}{\partial \dot{x}}-\frac{1}{\tau^2}\frac{\partial^2 f_0}{\partial \dot{x}^2}+\frac{2}{\tau^2}\frac{\partial f_0}{\partial \dot{x}}\frac{\partial f_1}{\partial \dot{x}}. \tag{81}$$

Equation (80) is of the Hamilton–Jacobi type. It can be checked by direct substitution into (80) that the minimal action $S(\dot{x},x)$ solves this equation, i.e. the identification $f_0(\dot{x},x)\equiv S(\dot{x},x)$ is verified. For a general potential it cannot be solved analytically. However, near the top of the barrier we may approximate $V(x)$ by the quadratic potential

$$V(x)=-x^2/2 \tag{82}$$

and look for a solution of the form

$$f_0(\dot{x}, x) = f_0(0, 0) + \alpha_0 x^2 + \beta_0 \dot{x}^2. \tag{83}$$

Substituting (82) and (83) into (80), one finds there are two possible sets of values for the coefficients α_0, β_0. Either

$$\alpha_0 = (\tau - 1)/2 \qquad \beta_0 = \tau(1-\tau)/2 \tag{84}$$

or $\alpha_0 = 0 = \beta_0$. For $\tau < 1$, the non-trivial solution (84) is physically reasonable: $f_0(\dot{x}, x)$ is at a maximum with respect to x at $x = 0$, corresponding to a minimum of the probability density at the top of the barrier, and $f_0(\dot{x}, x)$ is at a minimum with respect to $\dot{x}$ at $\dot{x} = 0$, which is consistent with the fact that the global minimum of the action at fixed x occurs at $\dot{x} = 0$. For $\tau > 1$, the solution (84) is unphysical and one is left with the trivial solution, $\alpha_0 = \beta_0 = 0$. However, it is possible to find a non-constant solution of (80), of the form (Bray *et al* 1990)

$$f_0(\dot{x}, x) = f_0(0, 0) + (\dot{x})^p G(x/\dot{x}) \tag{85}$$

where $p > 2$. Substituting (85) into (80), one finds that the nonlinear term $(\partial f_0/\partial \dot{x})^2$ is negligible for small x, $\dot{x}$. Omittiing this term, one is left with a linear differential equation for G which can be solved analytically to give

$$f_0(\dot{x}, x) = f_0(0, 0) + C[(\dot{x} - x)^\tau |\tau\dot{x} + x|]^{p/(1+\tau)} \tag{86}$$

where C is an undetermined constant. The value of p can be determined from the following argument in the regime $x < 0$, $\dot{x} > 0$: according to (86), the quantity $\delta f_0(\dot{x}, x) \equiv f_0(\dot{x}, x) - f_0(0, 0)$ vanishes along the line $\tau\dot{x} = -x$. On physical grounds this should be a simple zero of $\delta f_0(\dot{x}, x)$ which requires $p = 1 + \tau$ and thus, from (86),

$$f_0(\dot{x}, x) = f_0(0, 0) + C(\dot{x} - x)^\tau (\tau\dot{x} + x). \tag{87}$$

Summarizing the above results for $f_0(\dot{x}, x)$ near the top of the barrier in the regime $x < 0$, $\dot{x} > 0$ we have

$$f_0(\dot{x}, x) \simeq \begin{cases} f_0(0, 0) - (1-\tau)x^2/2 + \tau(1-\tau)\dot{x}^2/2 & \tau < 1 \\ f_0(0, 0) + C(\dot{x} - x)^\tau (\tau\dot{x} + x) & \tau > 1. \end{cases} \tag{88}$$

This result was originally obtained by Bray *et al* (1990). It explains the 'plateau' in the minimal action $S(0, x)$ which appears for $\tau > 1$ at the top of the potential barrier (see figure 2).

A similar analysis can be carried out for $f_1(\dot{x}, x)$ which is related, through (79), to the prefactor multiplying the exponential in the conditional joint probability distribution. For $\tau < 1$ it can be expanded as a power series of the form

$$f_1(\dot{x}, x) = f_1(0, 0) + \alpha_1 x^2 + \beta_1 \dot{x}^2 + \ldots . \tag{89}$$

whereas for $\tau > 1$ the appropriate solution does not have this simple structure. In order to solve (81) for $\tau > 1$ it will prove convenient to introduce the function

$$R(\dot{x}, x) \equiv \exp(-f_1(\dot{x}, x)) \tag{90}$$

which is proportional to the prefactor multiplying the exponential in (79). Changing variables from f_1 to R in (81), one obtains

$$0 = -\dot{x}\frac{\partial R}{\partial x} + \frac{1}{\tau}(1 + \tau V'')\left(R + \dot{x}\frac{\partial R}{\partial \dot{x}}\right) + \frac{V'}{\tau}\frac{\partial R}{\partial \dot{x}} - \frac{R}{\tau^2}\frac{\partial^2 f_0}{\partial \dot{x}^2} - \frac{2}{\tau^2}\frac{\partial f_0}{\partial \dot{x}}\frac{\partial R}{\partial \dot{x}}. \tag{91}$$

We expect to be able to obtain the solution of this equation by adopting a similar approach to that used in obtaining the zeroth order result f_0. In analogy with (85), we look for a solution of the form

$$R = \dot{x}^q H(x/\dot{x}). \tag{92}$$

Substituting this into (91) and using the form (82) for the potential at the top of the barrier, as well the previously obtained solution (86) for f_0, one finds that the last two terms are negligible compared with the others for small x, $\dot{x}$. By omitting these two terms, one obtains a linear equation in $H(x/\dot{x})$ which is *independent* of the zeroth order result f_0. Solving the linear equation for H and substituting the solution into (92) to obtain R, yields

$$R = C'(\dot{x}-x)^{[(\tau-1)(q+1)+q]/(1+\tau)}|x+\tau\dot{x}|^{[q\tau-(\tau-1)(q+1)]/(1+\tau)} \tag{93}$$

in the region $\dot{x}>0$, $x<0$. C' is an undetermined positive constant. We can determine the value of q using a similar argument to that used in determining p in (86). On physical grounds we require that the stationary probability does not vanish along the line $\tau\dot{x}=-x$. Examining equation (93) for the prefactor R, we see that his condition can only be met if the exponent,

$$\frac{q\tau-(\tau-1)(q+1)}{1+\tau} \tag{94}$$

is identically zero, which requires that $q=\tau-1$. Substituting this value of q into (93), one obtains

$$R = C'(\dot{x}-x)^{\tau-1}. \tag{95}$$

Having obtained the zeroth order and the first order results f_0 and f_1 for both $\tau<1$ and $\tau>1$, we are now in a position to write down the small-D form of the stationary joint probability distribution $P_{st}(\dot{x},x)$ in the region near the top of the potential barrier, for $\dot{x}>0$, $x<0$. Combining the results (88) and (95) using (79), we obtain for small x and $\dot{x}$

$$P_{st}(\dot{x},x) = N'(\dot{x}-x)^{\tau-1}\exp(-C(\dot{x}-x)^{\tau}(\tau\dot{x}+x)/D) \qquad \tau>1. \tag{96}$$

To compare the results of this section with the earlier path-integral analysis, we must first obtain the marginal distribution $P_{st}(x)$ from (96), by integrating over $\dot{x}$. For $D\to 0$, the integral can be investigated by steepest descents, but to do this we need to find the corresponding results to (96) for other values of x and $\dot{x}$. If $\dot{x}<0$, $x<0$, continuity with (87) at $\dot{x}=0$ requires that the exponent p in (86) is again equal to $(1+\tau)$ and that the arbitrary multiplying constant is again C. The singular behaviour when $\dot{x}=x$ is physical; in path-integral terms it corresponds to an interchange of dominant paths between those which are confined entirely to the region $x<0$ (which dominate when $\dot{x}<x$) and those involving the other well (which dominate when $\dot{x}>x$). A similar argument to that leading to (87) can be given when $x>0$, $\dot{x}<0$ and continuity at $\dot{x}=0$ will finally give us the form of $f_0(\dot{x},x)$ when $x>0$, $\dot{x}>0$. The generalization of (87) to all four quadrants is

$$f_0(\dot{x},x) = f_0(0,0) + C\,\mathrm{sign}(\dot{x}-x)|\dot{x}-x|^{\tau}(\tau\dot{x}+x) \tag{97}$$

where C is an undetermined positive constant. Similarly, since the stationary distribution must be positive, the generalization of (95) is

$$R(\dot{x},x) = C'|\dot{x}-x|^{\tau-1}. \tag{98}$$

Using (97) and (98) and performing the integral over $\dot{x}$ in the limit $D \to 0$, one finds

$$P_{st}(x) \simeq M'|x|^{(\tau-1)/2} \exp(C|x|^{\tau+1}/D) \qquad \tau > 1. \tag{99}$$

Equation (99) shows that $P_{st}(x)$ approaches zero as $|x|$ approaches zero and thus that the logarithm, $\ln P_{st}(x)$, diverges at $x=0$. This explains the numerical results for $\ln P_{st}(x)$ plotted in figure 10 using the results of the path-integral analysis. This graph indicates a change in behaviour at $\tau = 1$ which is consistent with a logarithmic divergence at $x=0$ for $\tau > 1$. The logarithmic divergence predicted by the small-D analysis is not, however, seen in the numerical simulations of the Langevin equation carried out at $D=0.1$, $\tau=3$, which provides evidence of the breakdown of the small-D expansion at the top of the barrier for $\tau > 1$.

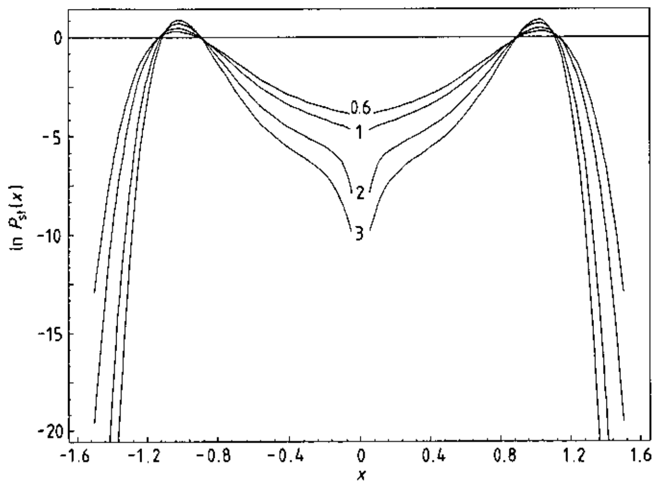

Figure 10. The value of $\ln P_{st}(x)$ from the path-integral analysis, plotted for $\tau = 0.6$, 1.0, 2.0 and 3.0.

We can understand this breakdown in more detail by analysing the expression for $P_{st}(x)$ as an integral over $\dot{x}$, without employing the method of steepest descents. Before doing this, however, it is useful to study the form of the next order correction to (99) for $\tau > 1$. If we write

$$P_{st}(\dot{x}, x) = [R(\dot{x}, x) + DQ(\dot{x}, x) + O(D^2)] \exp(-f_0(\dot{x}, x)/D) \tag{100}$$

then the function $Q(\dot{x}, x)$ satisfies the equation

$$0 = -\dot{x}\frac{\partial Q}{\partial x} + \frac{1}{\tau}(1+\tau V'')\left(Q + \dot{x}\frac{\partial Q}{\partial \dot{x}}\right) + \frac{V'}{\tau}\frac{\partial Q}{\partial \dot{x}} - \frac{Q}{\tau^2}\frac{\partial^2 f_0}{\partial \dot{x}^2} - \frac{2}{\tau^2}\frac{\partial f_0}{\partial \dot{x}}\frac{\partial Q}{\partial \dot{x}} + \frac{1}{\tau^2}\frac{\partial^2 f_0}{\partial \dot{x}^2}. \tag{101}$$

Assuming a scaling form $Q(\dot{x}, x) = \dot{x}^r I(x/\dot{x})$ as before, the nonlinear terms are again found to be negligible in the region of interest. However, the inhomogenuous term $(1/\tau^2)(\partial^2 f_0/\partial \dot{x}^2)$ is not negligible and in fact determines the exponent r to be $(\tau - 3)$. The differential equation for I is also more complicated because of this term, and cannot be solved in closed form. However, in the limit $\dot{x} \to 0$, Q can be determined, which is all that is required if we are to evaluate $P_{st}(x)$ to next order. The resulting expression is

$$P_{st}(x) \simeq M'|x|^{(\tau-1)/2} \exp(C|x|^{\tau+1}/D)[1 + O(D/|x|^{\tau+1}) + O(D/|x|^2) + O(D^2)]. \tag{102}$$

Thus there are two types of corrections to this order. The $O(D/|x|^{\tau+1})$ terms come from the small $\dot{x}$ expressions of f_0 and R, whereas the $O(D/|x|^2)$ terms come from Q evaluated at $\dot{x}=0$. For $\tau>1$, the former dominate the latter in the small $|x|$ limit. This suggests that if we truncate the expression for $P_{st}(\dot{x}, x)$, ignoring $O(D)$ corrections to $R(\dot{x}, x)$, that $P_{st}(x)$ would be a function of the combination $D/|x|^{\tau+1}$ only. Equation (102) is the $D\to 0$ form of this function and clearly breaks down for $|x| \leqslant D^{1/(\tau+1)}$. However, for $|x|\to 0$, so that $D/|x|^{\tau+1} \gg 1$, presumably a different expansion applies which gives a finite result when $x=0$. It is relatively straightforward to see this explicitly. From (97) and (98)

$$P_{st}(x)=\int_{-\infty}^{\infty} \mathrm{d}\dot{x}\, C'|\dot{x}-x|^{\tau-1} \exp(-S_c/D) \exp(-C \operatorname{sign}(\dot{x}-x)|\dot{x}-x|^{\tau}(\tau\dot{x}+x)/D) \quad (103)$$

where we have replaced $f_0(0,0)$, the instanton action for uphill paths, by the more compact notation S_c. This probability distribution is not yet normalized. Since, in the small-D limit, the normalization integral will be completely dominated by the region near the bottom of the potential wells, we can use the argument given in (36) *et seq* to show that a factor of D^{-1} must be included in (103) if $P_{st}(x)$ is to be correctly normalized. Using this and making the change of variable $\rho=\dot{x}-x$, changing the range of integration to $0\leqslant\rho<\infty$, and then making the further change of variable $z=(\tau C/D)^{\tau/(\tau+1)}\rho^{\tau}$ gives

$$P_{st}(x)=D^{-1/(\tau+1)} \exp(-S_c/D)\Phi(|x|/D^{1/(\tau+1)}) \quad (104)$$

where

$$\Phi(\lambda)=\tilde{C}' \int_0^{\infty} \mathrm{d}z \cosh[\tilde{C}\lambda z] \exp-[z^{(\tau+1)/\tau}] \quad (105)$$

where $\tilde{C}$ and $\tilde{C}'$ are constants. It is easy to check that evaluating Φ for $\lambda\to\infty$, i.e. $D\to 0$, leads to (102). However, if we take $|x|\to 0$, D fixed, then we see that

$$P_{st}(0) \sim D^{-1/(\tau+1)} \exp(-S_c/D) \quad (106)$$

a non-zero result, as expected.

From (38) we can determine $P_{st}(x)$ at the bottom of the potential well at $x=-1$; it equals $D^{-1/2}$, up to a constant. Hence we may also write (106) as

$$\frac{P_{st}(0)}{P_{st}(-1)} \sim D^{(\tau-1)/2(\tau+1)} \exp(-S_c/D). \quad (107)$$

This result is similar to one derived by Luciani and Verga (1987, 1988) for a piecewise-linear problem and using collective coordinate methods in a path-integral approach. However, in their result, the power of D in the prefactor has a different sign from that in (107). We do not understand this discrepancy, especially since our result is tied in via the crossover function Φ to the small-D behaviour (99) which agrees with numerical simulations.

7. Conclusion

In this paper we have shown that the path-integral formulation of model (1) with exponentially correlated noise can be used to obtain an expression (equation (67)) for the stationary probability distribution, $P_{st}(x)$, in the small-D limit. Each of the factors in this expression was evaluated for various values of the noise correlation time, τ, for the quartic bistable potential $V(x) = -x^2/2 + x^4/4$. A direct simulation showed this to be an excellent approximation for $D \leqslant O(0.1)$, except for a small region near the top of the barrier at $x = 0$, when $\tau > 1$. An analytic investigation of this phenomenon in section 6 revealed that the small-D approximation breaks down for values of x such that $|x| \leqslant D^{1/(\tau+1)}$, when $\tau > 1$. For these values of x we expect (67) to be inapplicable; the expression (106) for P_{st} right at the top of the barrier shows the strong non-analytic dependence on D which could not have been obtained by a straightforward steepest descent calculation.

The existence of a critical value of τ at which the nature of the stationary probability distribution changes has also been found in other studies (Hänggi *et al* 1989, Debnath *et al* 1990). The discussion presented here has the merit of being systematic and well controlled. We hope that we have shown the power of the path-integral approach to the calculation of the stationary probability distribution while at the same time emphasizing the fact that the evaluation of the path-integral by steepest descents is only valid in the limit $D \to 0$. This has already been beautifully illustrated by Mannella *et al* (1990) for the calculation of the mean first-passage time; this paper, then, is an attempt to do the same for the stationary probability distribution.

The techniques discussed here are, in principle, generalizable to other, more complex, systems. It would be interesting to know if the breakdown of the small-D expansion for a critical value of τ is a generic feature of stochastic processes where the noise is coloured, and if so, whether a simple characterization of this phenomenon is possible.

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